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Note

Sturm–Liouville Systems with Potentials 0.5 cos 2*nx*

1. INTRODUCTION

In this note we compare the eigenvalues of the Sturm–Liouville system $-u'' = \lambda u$ with those of $-u'' + (\frac{1}{2}\cos 2nx)u = \lambda u$; $n = 1, 2, \ldots$. The systems are assumed to be on a finite interval, and the eigenfunctions are assumed to have symmetrically located discontinuities at $d = \pi/m$ and $d = (m-1)\pi/m$ for $m = 3, 4, 5, \ldots$, satisfying symmetric jump conditions. We find that the differences between corresponding eigenvalues were quite small and showed a distinct, repetitive pattern after the nth eigenvalue. The nth eigenvalues of the two systems show the greatest difference. Furthermore, when $n = mk$ for $k = 1, 2, 3, \ldots$, the spectra of the two systems are almost identical except at the mkth eigenvalue. We implement an algorithm to solve the inverse Sturm–Liouville problem for the two systems to show that this mkth eigenvalue characterizes the difference between the two systems.

The eigenvalues of physical systems have been extensively studied by Andrew [1], Hochstadt [10], McNabb [14], Paine [15, 16], and others as they constitute the critical data for solving the associated inverse problem. Our motivation for examining the spectra of discontinuous Sturm–Liouville systems is to gain insight into the geophysical inverse problem; scientists seek to determine the density profile of the earth using eigenfrequency data from seismic waves (see Bolt [2] and Bolt and Uhrhammer [3]). A detailed discussion on the mathematical aspects of the seismological inverse problem is given in Hald [6–8].

2. TWO STURM–LIOUVILLE SYSTEMS

We consider the eigenvalues of the systems

System 1,

$$-u'' = \lambda u.$$

System 2,

$$-u'' + (0.5 \cos 2nx)u = \lambda u$$

on the interval $0 \leqslant x \leqslant \pi$ satisfying the symmetric boundary conditions

$$u(0) = u(\pi) = 1; \qquad u'(0) = u'(\pi) = 0$$

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with two symmetrically located discontinuities at $x = d_1 = d$ and $x = d_2 = \pi - d$; $0 < d_1 < \pi/2$ satisfying the jump conditions

$$u(d_1+) = au(d_1-), \qquad u'(d_1+) = a^{-1}u'(d_1-) + bu(d_1-),$$
$$u(d_2-) = au(d_2+), \qquad u'(d_2-) = a^{-1}u'(d_2+) - bu(d_2-).$$

In our experiment we set the jump constants for the eigenfunction and its derivative to be $a = 1.5$ and $b = 0.5$, respectively, and the discontinuities to be located at $d_1 = \pi/5$ and $d_2 = 4\pi/5$. The differences between the corresponding eigenvalues of System 1 and System 2 were less than 0.3 and showed a distinct, repetitive pattern after the nth eigenvalue where $n = 1, 2, 3, 4, 6, 7, 8, 9$. (See Fig. 1.) We also note that the nth eigenvalues of the two systems show the greatest difference. For the choices $n = 5$ and $n = 10$ we found that only a finite number of corresponding eigenvalues noticeably differ. In particular, the fifth and tenth eigenvalues show the greatest disparity. See Fig. 2 for the case $n = 5$. Data from the case $n = 10$ is similar, where the essential feature (i.e., a sharp downward peak) at eigenvalue 5 appears instead at eigenvalue 10.

To show that the few differing eigenvalues characterize the difference between the two systems, we implemented an algorithm to solve the inverse Sturm–Liouville problem [11, 12]. The algorithm assumes knowledge of the eigenvalues, jump and boundary conditions, and symmetric potential function of a Sturm–Liouville system. The eigenvalues from a system with an unknown, symmetric potential are given. The inverse problem is to determine this unknown potential. When the algorithm is implemented, the potential from the known system is perturbed towards the desired potential by replacing eigenvalues from the known system with those from the unknown system. As a greater number of eigenvalues are replaced, a better picture of the desired potential emerges. The algorithm is guaranteed to converge

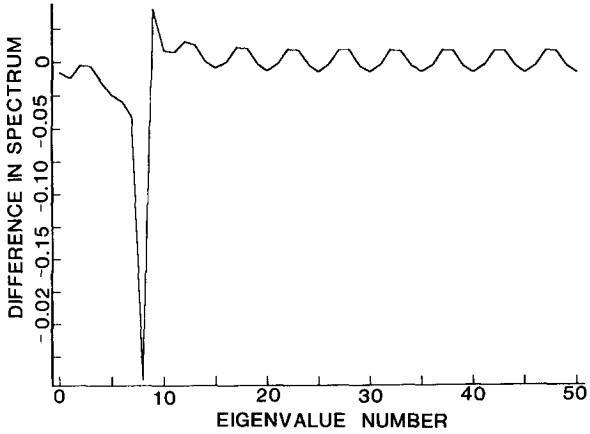

FIG. 1. $Q = 0$ vs $Q = 0.5 \cos 16X$.

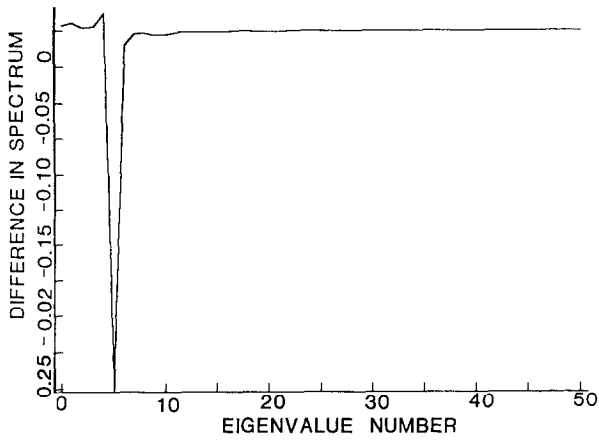

FIG. 2. $Q = 0$ vs $Q = 0.5 \cos 10X$.

so long as only a finite number of the corresponding eigenvalues in the known and unknown systems differ. This reconstruction algorithm is an extension of one by Hald [9] to solve the continuous, inverse Sturm–Liouville problem.

We chose the system with potential $q \equiv 0$ to be the initial system and that with potential $q = 0.5 \cos 2nx$ to be the target system. For $n = 5$, i.e., $q(x) = 0.5 \cos 10x$ the algorithm gives a potential very close to the zero potential when only the first $n - 1 = 4$ eigenvalues are used. (See Fig. 3.) This result is expected since the first four eigenvalues of the systems are very close. After the fifth eigenvalue is passed, i.e., $n + 1 = 6$ eigenvalues are used, the reconstruction yields an excellent approximation to the potential $q(x) = 0.5 \cos 10x$. These results can be seen by

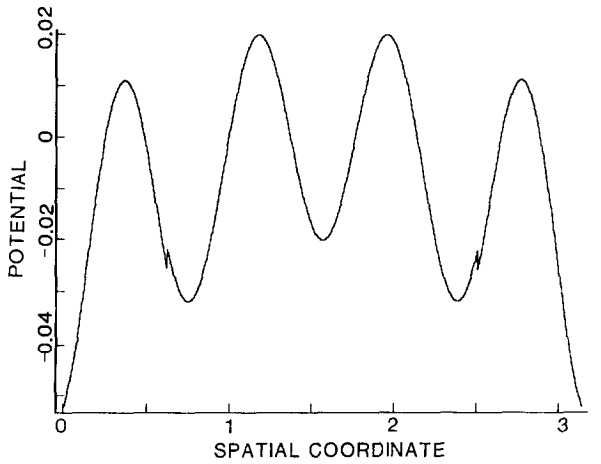

FIG. 3. $Q = 0.5 \cos 10X$ (5 eigenvalues).

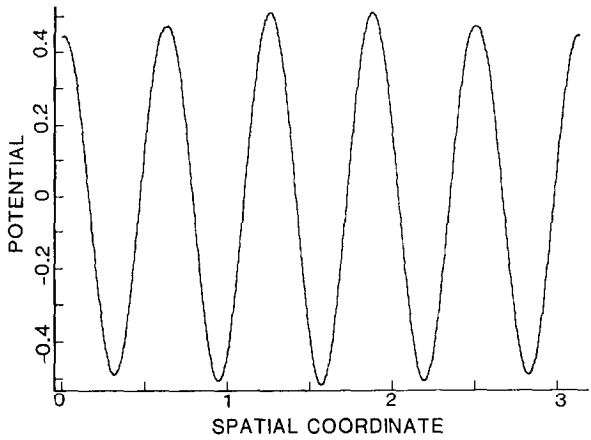

FIG. 4. $Q = 0.5 \cos 10X$ (6 eigenvalues).

examining the pictures of $q(x)$ or by considering the L_1, L_2 and L_∞ errors. (See Figs. 4 and 5 and Tables I–III below.)

Experiments similar to those for the choice $n = 5$ were run for the case $n = 10$. As expected the algorithm gives a potential very close to the zero potential when only the first $n - 1 = 9$ eigenvalues are used. After the tenth eigenvalue is passed, i.e., $n + 1 = 11$ or more eigenvalues are used, the reconstruction is an excellent approximation to the potential $q(x) = 0.5 \cos 20x$.

Experiments to study the effect of the discontinuity on the eigenvalues were conducted when the discontinuity is placed at $d = 1/m$ for $m = 3, 4, 5, \ldots$. The difference in the eigenvalues of systems 1 and 2 was calculated. A sharp peak (i.e., large difference) can be observed at the nth eigenvalue, and an oscillatory pattern can be

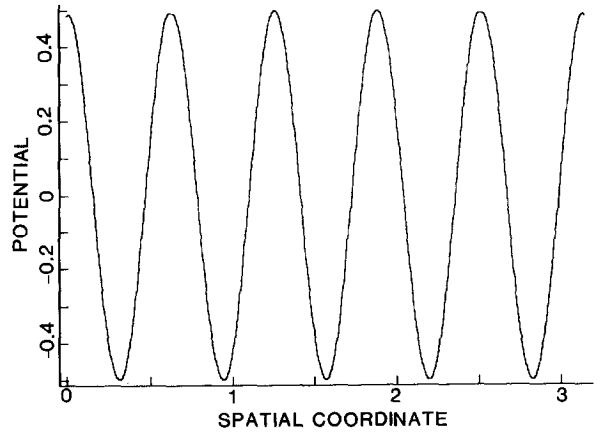

FIG. 5. $Q = 0.5 \cos 10X$ (15 eigenvalues).

TABLE I

Eigenvalues of S-L Systems ($a = 1.5$, $b = 0.5$, $d = \pi/5$)

eigenvalue	$q(x) = 0$	$q(x) = 0.5 \cos 10x$	$q(x) = 0.5 \cos 20x$
0	0.25453517	0.25089171	0.25362842
1	1.8966688	1.8914568	1.8954365
2	4.7043829	4.7021128	4.7038985
3	8.1565250	8.1537540	8.1560155
4	14.594351	14.582115	14.592846
5	25.542353	25.790359	25.540129
6	39.235588	39.244831	39.233435
7	51.254197	51.255970	51.253243
8	61.607223	61.608499	61.606007
9	77.279468	77.281955	77.273747
10	100.54482	100.54766	100.79431
11	126.55437	126.55567	126.55933
12	147.80341	147.80379	147.80439
13	165.05804	165.05838	165.05876
14	189.96170	189.96250	189.96316

TABLE II

Reconstruction of $Q = 0.5 \cos 10X$
($a = 1.5$, $b = 0.5$, $d = \pi/5$, grid $= \pi/10000$)

eig	L_1 error	L_2 error	L_∞ error
0	0.999	0.627	0.508
1	0.999	0.626	0.518
2	0.999	0.626	0.522
3	1.000	0.627	0.528
4	1.009	0.633	0.552
5	0.0404	0.0273	0.0563
6	0.0175	0.0136	0.0378
7	0.0166	0.0140	0.0342
8	0.0168	0.0130	0.0317
9	0.0134	0.00963	0.0267
10	0.00675	0.00563	0.0210
11	0.00570	0.00491	0.0184
12	0.00537	0.00505	0.0177
13	0.00569	0.00480	0.0170
14	0.00458	0.00384	0.0154

TABLE III

Reconstruction of $Q = 0.5 \cos 20X$
($a = 1.5$, $b = 0.5$, $d = \pi/5$, grid $= \pi/10000$)

eig	L_1 error	L_2 error	L_∞ error
0	1.00	0.627	0.502
1	1.00	0.627	0.505
2	1.00	0.627	0.506
3	1.00	0.627	0.506
4	1.00	0.627	0.509
5	1.00	0.627	0.514
6	0.999	0.626	0.518
7	0.999	0.626	0.520
8	1.00	0.627	0.522
9	1.00	0.630	0.534
10	0.0214	0.0146	0.0348
11	0.00965	0.00775	0.0249
12	0.00901	0.00805	0.0230
13	0.00943	0.00744	0.0215
14	0.00696	0.00546	0.0186

seen for eigenvalues greater than n. As in the example above, when $n = m \cdot k$ for $k = 1, 2, 3, \ldots$, the eigenvalues of the two systems are close except at the mkth eigenvalue. Furthermore, results from implementing the numerical algorithm in [11] can be predicted; the algorithm gives a potential very close to the zero potential when the first $(mk - 1)$ eigenvalues are used and yields an excellent approximation to the potential $q(x) = 0.5 \cos 2nx$ when the mkth eigenvalue is used.

Joyce McLaughlin of Rensselaer Polytechnic Institute has suggested that our observations might be explained by examining the asymptotic expansions of the eigenvalues. Expansions for the eigenvalues of continuous Sturm–Liouville systems have been studied by Borg [4] and Hochstadt [10]. A general expression for the expansion for systems with discontinuous eigenfunctions cannot exist, as shown by Hald [5]. A general technique for some special cases has been developed for the single discontinuity case [13]. We hope that our experimental results will give some insight into this area.

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MEI KOBAYASHI

IBM Research, IBM Tokyo Research Laboratory
5-19, Sanbancho, Chiyoda-ku, Tokyo 102, Japan